Longitudinal Dispersion Mechanisms in Packed Beds

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A theoretical study has been made of longitudinal dispersion mechanisms during steady flow of a fluid through unconsolidated spherical beads. The mathematical model utilizes a step function input of thermal energy and presents the solution for the transient behavior of the system. The longitudinal dispersion of the step input is considered the result of eddy mixing of the fluid, molecular conduction within the fluid, and a finite time lag for heat transfer to occur between fluid and particle. The latter mechanism is characterized by both a fluid film resistance and an intraparticle resistance. The exact solution, involving an infinite integral, is approximated to an error function form. The character of the contribution of each mechanism is apparent from the form of the approximate solution, and furthermore, the contributions are shown to be additive to yield the total dispersion. The conditions for convergence to the approximate solution are indicated. The equations are presented such that the analysis is applicable to both heat and mass transfer processes.

REVIEW OF LITERATURE

Theoretical studies of longitudinal dispersion of either thermal energy or component concentration in fixed-bed systems are readily available in the literature, and the analogy between heat conduction and component diffusion renders the analyses interchangeable. Generally, there are two dispersion mechanisms considered:

1. The fluid phase conduction or diffusion is characterized by a dispersion coefficient containing the effect of eddy mixing of the fluid as it flows through the interstices of the pack and the effect of molecular conduction or diffusion within the fluid.
2. The finite time lag required for transfer between fluid and particle which is considered as two distinct steps, transport across a stagnant fluid film surrounding the particles, and transport within the particles characterizing the time required to dissipate intraparticle gradients.

Specifically, Van Deemter et al. (*17*) modified the mass transfer work of Lapidus and Amundson (*12*), approximating their general solution by a Gaussian solution in which the variances due to mechanism 1 and the first step of mechanism 2 were found to be additive. Klinkenberg and Sjenitzer (*10*) also proposed the idea of additive variances for the different mechanisms. Green (*7, 8*), based on these works, presented approximate solutions for the case which included mechanisms 1 and the first step of mechanism 2 plus the additional mechanism of longitudinal conduction in the solid phase. Heat transfer data were taken and compared with the theoretical model (*7*).

Rosen (*15, 16*) studied the combined effect of mechanism 2 for a concentration step input. His work demonstrates the terms to be additive when the exact solution is reduced to an approximate error function solution. Kasten et al. (*11*) made an independent study of the same mechanism in which a distinction was made between interparticle fluid and intraparticle fluid.

Deisler and Wilhelm (*4*) studied all of the abovementioned mechanisms by utilizing the steady state frequency response of a cosine concentration input. The form of the approximate expression presented for the amplitude attenuation factor shows the individual contributions of the various mechanisms to be additive. Liles and Geankoplis (*13*), among others, have utilized frequency response methods to determine mass dispersion coefficients for liquids. McHenry and Wilhelm (*14*) have made a similar study using gases. These coefficients represent dispersion due to mechanism 1 only. In these studies, transfer between fluid and particle does not occur, and at high velocity, dispersion is essentially due to eddy mixing of the fluid. Gottschlich (*9*) has made a study in which the eddy mixing is subdivided into interstitial velocity effects and the capacitance effect of a stagnant fluid film. In this manner, Gottschlich is able to explain the difference between experimental eddy mixing dispersion coefficients for liquids and gases. Glaser (*5, 6*) and his co-workers also suggest subdividing the eddy mixing in this manner and discuss the relative contribution of each effect. Their work is primarily oriented toward two-phase flow.

This paper describes a means of determining the exit profile of a packed bed in which the longitudinal dispersion of the step input is considered the result of all of the above listed dispersion mechanisms. An exact solution is presented in the form of an infinite integral which, under certain conditions, reduces to an approximate error function solution. This approximation clearly shows the state of additivity in the argument of the error function. The conditions for convergence to the approximate solution are discussed, and figures are presented comparing it with the exact solution for several different sets of conditions. Heat transfer data analyzed by means of this approximate solution are presented elsewhere (*1*).

MATHEMATICAL MODEL

The differential equations presented by Deisler and Wilhelm (*4*) in their mass transfer work are applicable to the analogous heat transfer problem:

$$\frac{\partial C}{\partial t} + V\frac{\partial C}{\partial z} - D_L\frac{\partial^2 C}{\partial z^2} = -\,(\partial \overset{\Delta}{q}/\partial t)/m \qquad (1)$$

$$\partial q_i/\partial t = D_s\left[\,\partial^2 q_i/\partial r^2 + \frac{2}{r}\,\partial q_i/\partial r\right] \qquad (2)$$

The various C and q terms are analogous to concentrations, since by definition they represent energy per unit volume.

Equation (1) results from conservation of energy within a differential segment of fluid at some position z from the bed entrance. Equation (2) results from conservation of energy within a differential spherical shell located at some radius r from the center of a spherical particle. The following major assumptions apply:

1. Homogeneous, unconsolidated pack.
2. Uniform, flat velocity profile.
3. Negligible radial heat losses from the packed bed.
4. Negligible solid-interparticle conduction through point-to-point contact.
5. Physical properties of both phases are temperature independent for any given set of conditions.
6. Particle diameter is small enough so that the temperature drop across a single particle due to longitudinal fluid gradients does not influence the temperature distribution within the particle, that is, radial symmetry exists within the individual spherical particles.

The boundary conditions used are:

Equation (1):

$C(z, 0) = 0$

$C(0, t) = C_I \equiv \rho_w c_w (T_{wI} - T_{w0})$

$C(\infty, t) = 0$

Equation (2):

$q_i(r, z, 0) = 0; \; z \geq 0$

$q_i(0, z, t) \neq \infty; \; z \geq 0$

$q_i(b, z, t) \equiv q_s(z, t)$ given by

$$\frac{\partial \overset{\Delta}{q}}{\partial t} = \frac{3h}{b \rho_w c_w} (C - q_s/K); \; z \geq 0 \tag{3}$$

The last boundary condition [Equation (3)] is the link between Equations (1) and (2). It states mathematically that the rate of energy entering or leaving the solid particles must equal the rate of energy transported across a stagnant fluid film surrounding the particle surface. The latter transport is represented by the conventional expression, $ha(T_w - T_s)$, where T_s is the surface temperature of the particles. This procedure assumes that no energy accumulation occurs within the stagnant fluid film.

EXACT SOLUTION

The simultaneous solution for Equations (1) and (2) with the boundary conditions indicated involves the following steps. The details are presented elsewhere (*1*).

1. The solution for Equation (2) is available (*2, 16*) in a form that expresses the concentration distribution within the solid particles as a function of the variable surface concentration, $q_s(z, t)$.
2. This expression is first averaged by integration (*4, 16*) over the entire volume of the particle and then differentiated with respect to time to yield an expression for $\partial \overset{\Delta}{q}/\partial t$, the rate of change of the average energy concentration within the solid.
3. Next, the expression derived for $\partial \overset{\Delta}{q}/\partial t$ is introduced into Equation (1), and the Laplace transform is taken with respect to time.
4. When one makes use of the convolution theorem, uses Equation (3) in the Laplace domain to express the surface concentration $q_s(z, t)$ in terms of $\partial \overset{\Delta}{q}/\partial t$, expresses the result as the left-hand side of Equation (1), and substitutes the entire group back into the transformed equation, an ordinary, second-order linear differential equation results, having the transformed variable $\bar{C}$ as the dependent variable.
5. Solving the ordinary differential equation, applying the binomial theorem, and taking the inverse transform (*1, 16*), one obtains the following solution involving an infinite integral:

$$u(z, \theta) = \frac{1}{2} + \frac{2}{\pi} \int_0^\infty e^{-\left\{\gamma x H_1 + \frac{zD_L}{V^3}\left[\sigma^2\lambda^4 + \frac{2\sigma\lambda^2 H_2 \gamma}{m} \frac{\gamma^2}{m^2}(H_2^2 - H_1^2)\right]\right\}} \cdot \sin\left\{\sigma\theta\lambda^2 - \gamma x H_2 + \frac{zD_L}{V^3} \left(\frac{2\sigma\lambda^2 H_1 \gamma}{m} + \frac{2H_1H_2\gamma^2}{m^2}\right)\right\} \frac{d\lambda}{\lambda} \tag{4}$$

where H_1 and H_2 are complicated hyperbolic functions of λ and ν:

$$H_1(\lambda, \nu) = \frac{H_{D1} + \nu(H^2{}_{D1} + H^2{}_{D2})}{(1 + \nu H_{D1})^2 + (\nu H_{D2})^2} \tag{5}$$

$$H_2(\lambda, \nu) = \frac{H_{D2}}{(1 + \nu H_{D1})^2 + (\nu H_{D2})^2} \tag{6}$$

H_{D1} and H_{D2} are defined as

$$H_{D1} = \left[\lambda\left(\frac{\sinh 2\lambda + \sin 2\lambda}{\cosh 2\lambda - \cos 2\lambda}\right) - 1\right] \tag{7}$$

$$H_{D2} = \lambda\left(\frac{\sinh 2\lambda - \sin 2\lambda}{\cosh 2\lambda - \cos 2\lambda}\right) \tag{8}$$

The infinite integral of Equation (4) may be numerically integrated with a digital computer by simply calculating $f(\lambda)$ as a function of λ and by using Simpson's rule to evaluate the summation. The results of this type of integration for several different sets of conditions are presented in Figure 1. Rosen (*15*) used a more sophisticated method of numerical integration in his treatment of the case in which $D_L = 0$.

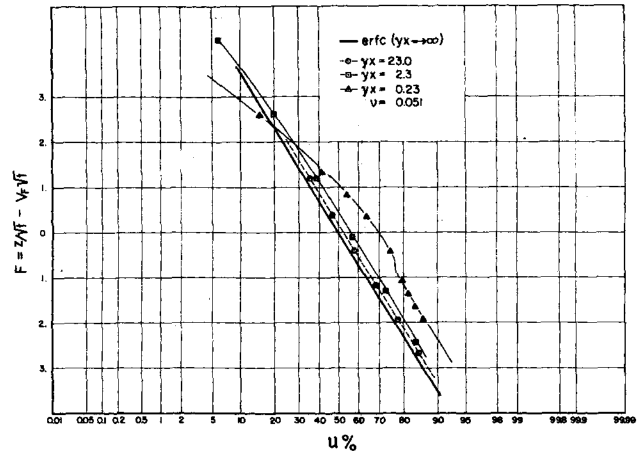

Fig. 1. Profile curves.

APPROXIMATE SOLUTION

Rosen (*16*) has tabulated exact values of H_{D1} and H_{D2} as a function of λ. If the expressions for H_{D1} and H_{D2} are expanded in a Maclaurin series as proposed by Deisler (*3, 4*), the following limiting expressions may be obtained:

$$H_{D1} = \frac{4\lambda^4}{45} \tag{9}$$

$$H_{D2} = \frac{2\lambda^2}{3} \tag{10}$$

Limiting values obtained from these expressions compare within 4% of the exact values obtained by Rosen if $\lambda \leq 1.0$.

When these limiting expressions for H_{D1} and H_{D2} are used, the following approximations are valid for H_1 and H_2 if the product $\lambda^4\nu^2$ is also small:

$$H_1(\lambda, \nu) = \frac{4\lambda^4(1 + 5\nu)}{45} \qquad (11)$$

$$H_2(\lambda, \nu) = \frac{2\lambda^2}{3} \qquad (12)$$

When one substitutes Equations (11) and (12) into Equation (4) and discards all λ terms of order 6 (λ^6) or higher, the infinite integral is simplified to the form

$$u(z, \theta) = \frac{1}{2} + \frac{2}{\pi}\int_0^\infty e^{-\lambda^4\left\{(4/45)\gamma x(1 + 5\nu) + \frac{zD_L}{V^3}\left(\sigma + \frac{2\gamma}{3m}\right)^2\right\}} \cdot \sin\{\sigma\theta\lambda^2 - (2/3)\gamma x\lambda^2\}\frac{d\lambda}{\lambda} \qquad (13)$$

Equation (13) may be reduced to an erf[*] form as follows:

Let

$$\xi^2 = \lambda^4\left\{(4/45)\gamma x(1 + 5\nu) + \frac{zD_L}{V^3}\left(\sigma + \frac{2\gamma}{3m}\right)^2\right\} \qquad (14)$$

and

$$y = \frac{\sigma\theta - (2/3)\gamma x}{2\left\{(4/45)\gamma x(1 + 5\nu) + \frac{zD_L}{V^3}\left(\sigma + \frac{2\gamma}{3m}\right)^2\right\}^{1/2}} \qquad (15)$$

Then

$$u(z, \theta) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty e^{-\xi^2}\sin 2\xi y\frac{d\xi}{\xi} = \frac{1}{2}(1 + \operatorname{erf} y) \qquad (16)$$

reduces to the following expression:

$$u(z, \theta) = \frac{1}{2}\left\{1 - \operatorname{erf}\left[\frac{\left(\frac{2}{3}\right)\gamma x - \sigma\theta}{2\left((4/45)\gamma x(1 + 5\nu) + (zD_L/V^3)\left[\sigma + \frac{2\gamma}{3m}\right]^2\right)^{1/2}}\right]\right\} \qquad (17)$$

If the argument of the erf in Equation (17) is multiplied top and bottom by $(b^2V_F/2D_s\sqrt{t})$, if $x = (z/mV)$, $\theta = t - \frac{z}{V}$ is substituted, and if z is replaced by V_Ft everywhere but in the numerator,[†] Equation (17) becomes

$$u(z, t) = \frac{1}{2}\left\{1 - \operatorname{erf}\left[\frac{(z/\sqrt{t}) - V_F\sqrt{t}}{2\sqrt{D_e}}\right]\right\} \qquad (18)$$

where

$$D_e = \frac{\rho_w c_w \phi D_L + \frac{[\rho_s c_s(1-\phi)V_F d_p]^2}{60k_s(1-\phi)} + \frac{[\rho_s c_s(1-\phi)V_F]^2}{ha}}{\rho_w c_w \phi + \rho_s c_s(1-\phi)} \qquad (19)$$

Using the following definitions

$$k_{s(ha)} = \frac{[\rho_s c_s(1-\phi)V_F d_p]^2}{60k_s(1-\phi)} \qquad (20)$$

$$k_{ha} = \frac{[\rho_s c_s(1-\phi)V_F]^2}{ha} \qquad (21)$$

$$D_L = (k_{wm}\phi + k_w\phi)/\rho_w c_w \phi \qquad (22)$$

one gets

$$D_e = \frac{k_{wm}\phi + k_w\phi + k_{ha} + k_{s(ha)}}{\rho_w c_w \phi + \rho_s c_s(1-\phi)} \qquad (23)$$

Equation (18) shows that the transient response to the step input is a dispersed step in which the degree of dispersion is characterized by an overall dispersion coefficient D_e. Equation (23) shows D_e to be the algebraic sum of the individual contributions of eddy mixing of the fluid ($k_{wm}\phi$), molecular conduction in the fluid $k_w\phi$, stagnant film resistance k_{ha}, and intraparticle resistance $k_{s(ha)}$.

CONDITIONS FOR CONVERGENCE

Three conditions are required in approximating the infinite integral of Equation (4) by the more explicit integral of Equation (13).

1. The value of λ must remain small ($\lambda \leq 1$) as required by Equations (9) and (10), and also in discarding the higher-ordered λ terms preceding Equation (13).
2. The product $\lambda^4\nu^2$ must remain small ($\lambda^4\nu^2 \leq 0.2$)[‡] as required by Equations (11) and (12).
3. All holding times must remain near the mean holding time z/V_F, as noted in the footnote of Equation (18).

Mathematically, the first two conditions require that the integrand of Equation (4) rapidly converges to zero and thus allows the infinite integral to be replaced by

$$\int_0^\infty f(\lambda)d\lambda = \int_0^\epsilon f(\lambda)d\lambda \qquad (24)$$

where $f(\lambda)$ represents the integrand of Equation (4), and where ϵ is set by either $\epsilon \leq 1.0$ or $\epsilon^4\nu^2 \leq 0.2$, depending upon the size of ν.

The function $f(\lambda)$ is the product of an exponential decaying function and a periodic sine function. The total function is thus a decaying sine wave in which both the period of oscillation and the degree of decay are functions of the system parameters.

If the decaying function reduces the function $f(\lambda)$ to negligible values for $\lambda > 1.0$, then the contribution to the integral is contained entirely in the function space $0 < \lambda < 1.0$. This being the case, the function $f(\lambda)$ may be replaced by an approximate function valid for small values of λ, resulting in Equation (13) where it is understood that the contribution to the integral for values of $\lambda > 1.0$ is negligible.

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[*] erf denotes the error function. For a discussion concerning its definition and properties see reference 2. Notice that $\operatorname{erf}(-y) = -\operatorname{erf}(y)$.

[†] Replacing z by V_Ft everywhere but in the numerator requires that all holding times are near the mean holding time, z/V_F. This is equivalent to saying that the dispersion zone length is small compared with the total length of the bed.

[‡] Equations (11) and (12) require that

$$\left(1 + \frac{4\lambda^4\nu}{45}\right)^2 + \frac{4\lambda^4\nu^2}{9} \to 1.0$$

For small values of λ the λ^4 term insures this convergence; however, for values of λ approaching 1.0, the added restriction of small values of ν must be added. The condition $\lambda^4\nu^2 \leq 0.2$ was arbitrarily set by limiting the error to 9%, that is

$$\left(1 + \frac{4\lambda^4\nu}{45}\right)^2 + \frac{4\lambda^4\nu^2}{9} \leq 1.09$$

The parameter γx, the leading exponential term in Equation (13), is the most critical parameter of the decaying function* and may be used as a criterion for the above conditions. The larger the value of γx, the more rapidly the $f(\lambda)$ function approaches zero, and thus the approximate solution becomes increasingly accurate. Notice that the parameter ν plays a double role. A large value of ν decreases the accuracy of Equations (11) and (12), while on the other hand it enhances convergence owing to its role in the decay function of Equation (13). Also, notice that high conductivity beads (metal) possess large values of ν and also large values of γx. This again enhances convergence owing to the strong role γx plays in the decaying function. For example a system of 6-mm. glass beads in water has a value of ν of 0.20 and a value of γx of 1.4 at a velocity of 100 ft./hr. In comparison, a system of 6-mm. lead beads in water has a value of ν of 6.5 and a value of γx of 43 also at a velocity of 100 ft./hr. Figure 2 demonstrates that $f(\lambda)$ converges much more rapidly for a system having a value of γx of 43 compared with a value of γx of 1.4.

The condition of the large γx is a restatement of condition 3 and has physical meaning. The parameter γx has been termed *effective bed length* by Rosen (*16*). If a simplified approach is taken, it may be considered as the product of two ratios. The first of these ratios is the mean bed holding time of the fluid, z/V, compared with the time required to eliminate intraparticle gradients, $(b^2/3D_s)$. The second ratio is the capacity of the solid phase to release heat $\rho_s c_s(1-\phi)$, compared with the capacity of the fluid phase to accept it, $\rho_w c_w \phi$. If the mean bed holding time is large, that is, γx is large, the early segments of fluid of the step function assume the previous temperature of the bed before they reach the bed exit, and thus the erf profile forms in the bed and arrives at the exit at some later time. However, if the mean bed holding time is short, the early segments of fluid do not have time to assume the previous temperature of the bed, and the erf profile does not stabilize. Under these latter conditions, the approximate solution no longer accurately describes the system, and the exact solution must be used.

Figure 1† demonstrates the effect of convergence on the accuracy of the approximate solution. This is done by presenting the exact solution for several different values of γx on arithmetic probability paper. In generating these curves, the mean bed holding time z/V is the only parameter varied.

The heavy solid line represents the erf solution under the assumption that the mechanisms are additive. As γx decreases, the departure from the approximate solution increases. For $\gamma x = 23$, the approximation is quite good except for a slight time lag. The slope of the two curves are equal, indicating the same degree of dispersion.‡ The time lag merely indicates a difference in the arrival time of the exit profile. The time lag originates when the argument of the sine term in Equation (4) is approximated by only the first two terms; that is, $\sigma\theta - \dfrac{2}{3}\gamma x$. This may be demonstrated by noticing that the integral of Equation (13) becomes zero $\left[u(z,\theta) = \dfrac{1}{2}\right]$ for $\theta = \left(\dfrac{2}{3}\right)$ $(\gamma x/\sigma)$, but the integral of Equation (4) becomes zero for

$$\theta = \frac{\gamma x H_2 - \dfrac{zD_L}{V^3}\left(\dfrac{2\sigma\gamma H_1\lambda^2}{m} + \dfrac{2H_1H_2\gamma^2}{m^2}\right)}{\sigma\lambda^2} \qquad (25)$$

If the limiting expressions for H_1 and H_2 are used, this becomes

$$\theta = \left(\frac{2}{3}\right)(\gamma x/\sigma) - \left(\frac{16}{135}\right) \left(\frac{zD_L}{V^3}\right)\left(\frac{\gamma}{m}\right)(1+5\nu)\left(1+\frac{\gamma}{m}\right)\lambda^4 \qquad (26)$$

The second term in Equation (26) causes the time shift. Notice that owing to the λ^4 term it becomes important only if the decay function has not sufficiently dampened the sine wave for values of $\lambda > 1.0$. At the value of $\gamma x = 0.23$, the exit profile is no longer of the erf form, and the approximate solution breaks down completely. The profile curve does appear to approach the erf form (straight line on probability paper) as u approaches 1.0.

Figure 2 presents the form of the $f(\lambda)$ function for the cases presented in Figure 1. The area under these curves is related to the temperature fraction by

$$u(z,\theta) = \frac{1}{2} + \frac{2}{\pi}\,(\text{area}) \qquad (27)$$

Thus, at small times t, the net area under the curves is negative giving a value of $u(z,\theta) < 0.5$. At $u(z,\theta) = 0.5$, the net area under the curve becomes zero, and for $u(z,\theta) > 0.5$, the area is positive. This is the case for all of the curves in Figure 2. Notice that for $\gamma x = 23$, the function converges very quickly for $\lambda > 1.0$, while for $\gamma x = 0.23$, the function does not converge until $\lambda > 8.0$. These convergence criteria agree qualitatively with the numerical work of Rosen (*15*) in his treatment of the case of $D_L = 0$.

The approximate solution has been used to evaluate experimental heat transfer data for a variety of physical systems. The details of these studies are presented elsewhere (*1*, *7*). The total dispersion was experimentally measured, and the dispersion due to fluid mixing k_{wm} was then determined by subtraction using Equation (23). The value of the mixing dispersion so determined was then compared for the various systems. No significant deviations were noticed in the mixing dispersion for 6-mm. metal and glass spheres, γx ranging from 1.4 to 83 for

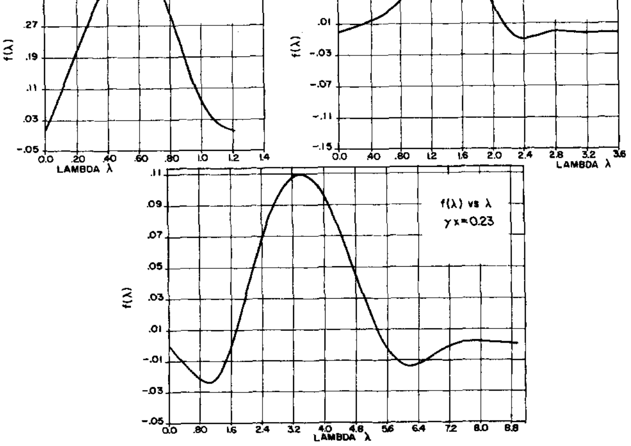

Fig. 2. Convergence curves.

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* The second term becomes negligible at high velocity.

† These curves were obtained by numerically integrating the exact solution with an IBM-7040 digital computer.

‡ The amount of dispersion is a sole function of the slope of the F vs. u curve (*7*).

these systems.* However, with 6-mm. Lucite spheres ($\gamma x = 0.50$) used, a significant increase in mixing dispersion was noticed. This can be explained by the fact that the approximate solution is no longer valid, and thus the mechanisms are no longer additive.

The experimental systems of Deisler (3, 4) may also be investigated in which γx varies from 2.4 to 51.2. Thus the approximate solution and the state of additivity should be valid for Deisler's systems. However, the velocities used by Deisler were very small ($V < 1.0$ ft./hr.), and, therefore, this would lead one to conclude that the conditions for convergence are more readily met for heat transfer systems than for mass transfer systems.

CONCLUSIONS

The following conclusions are drawn from the results of this investigation:

1. The longitudinal dispersion of a step function input may be adequately described by a total dispersion coefficient D_e if certain physical conditions are met. The conditions stem from the mathematical approximation of the infinite integral [Equation (4)] by an erf function [Equation (18)]. The convergence criteria for this approximation are expressed in terms of the parameter γx. A large value of γx enhances convergence. Physically, this means that the approximate solution is valid for long beds, operating at low fluid velocity, and packed with small, highly conductive particles.
2. The total dispersion coefficient D_e is composed of the algebraic sum of four individual dispersion coefficients each representing a dispersion mechanism [Equations (20), (21), and (22)].
3. If physical conditions are such that the approximate erf solution is not valid, the infinite integral may be numerically integrated to yield the exit profile. Results of numerical integration presented herein indicate that γx must be greater than 2.0 for the approximate solution to adequately describe the exit profile.

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NOTATION

a = particle area per unit volume of bed, $6(1-\phi)/d_p$, ft.
b = particle radius, in. or ft.
c = heat capacity, B.t.u./(lb.$_m$)(°F.)
C = change from the initial state of the fluid phase sensible heat; $\rho_w c_w (T_w - T_{w_o})$, B.t.u./cu. ft.
D = molecular diffusivity (mass or heat transfer), sq. ft./hr. $D = D_w$ unless otherwise stated
D_e = effective thermal diffusivity of porous medium
D_L = total diffusivity of the fluid phase, $E + D$, sq. ft./ hr.
d_p = solid particle diameter, in. or ft.
E = eddy mixing coefficient (mass or heat transfer), sq. ft./hr.
F = $z/\sqrt{t} - V_F\sqrt{t}$
h = heat transfer coefficient, B.t.u./(hr.)(sq. ft.)(°F.)
H_1 see Equation (5)
H_2 see Equation (6)
H_{D1} see Equation (7)
H_{D2} see Equation (8)
k = molecular thermal conductivity, B.t.u./(hr.)(ft.)(°F.)
k_e = effective thermal conductivity of porous media, B.t.u./(hr.)(ft.)(°F.)
$k_{(ha)}$ = effective coefficient characterizing a finite heat transfer rate between the solid and fluid phases controlled by a fluid film around the particles, B.t.u./(hr.)(ft.)(°F.)
$k_{s(ha)}$ = effective coefficient characterizing a finite time required to damp out temperature gradients within the solid particles, B.t.u./(hr.)(ft.)(°F.)
k_{wm} = eddy mixing dispersion coefficient, B.t.u./(hr.)(ft.)(°F.)
K = thermal equilibrium constant, $\rho_s c_s/\rho_w c_w$
m = ratio of void fraction to solid bed fraction, $\phi/1-\phi$
q = change from the initial state of the solid phase sensible heat; $\rho_s c_s (T_s - T_{s_o})$, B.t.u./cu. ft.
R = real part as opposed to imaginary part
R_f = film resistance, $b\rho_w c_w/3h$, hr.
s = Laplace transform variable, $\alpha + i\beta$
T = temperature, °F.
t = time, min. or hr.
u = temperature fraction, $T - T_o/T_I - T_o$
V = interstitial fluid velocity, ft./hr.
V_F = velocity of square heat front, $V_F = \rho_w c_w \phi V/\rho_w c_w \phi + \rho_s c_s (1-\phi)$
x = z/mV, hr.
z = longitudinal distance, ft.

Greek Letters

α = real part of complex Laplace transform variable s
β = imaginary part of complex Laplace transform variable s
ϵ = value of λ at which the function $f(\lambda)$ approaches zero
γ = $3D_s K/b^2$, 1/hr.
λ = variable of integration
$f(\lambda)$ = integrand of Equation (4)
ρ = density, lb.$_m$/cu. ft.
ϕ = porosity, void fraction
θ = $t - z/v$, hr.
σ = $2D_s/b^2$, 1/hr.
ν = $\gamma R_f = k_s/bh$

Subscripts

I = inlet condition
i = inside solid particles
o = original condition
s = solid phase
w = fluid phase
wm = fluid mixing

Superscripts

Δ = volume averaged variable
— = Laplace transform domain

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* For a packed section 1 ft. long and a fluid velocity of 100 ft./hr.

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Theoretical Study of Bubble Dynamics in Purely Viscous Fluids

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This paper analyzes the growth or collapse of a spherical bubble in an incompressible, viscous fluid. Theoretical results include the timewise variations in the bubble size and its growth or collapse rate, the fluid pressure, and the rate of energy dissipation. The analysis is general and may be applied to both Newtonian and non-Newtonian fluids. A comparison is given for the collapse of the bubble in several viscous fluids.

In connection with studies on cavitation and cavitation damage, it is desirable to have mathematical expressions for the pressure and velocity fields in the neighborhood of a growing or collapsing gas- or vapor-filled cavity in a liquid. The problem for a spherical bubble in an incompressible nonviscous liquid has been solved (*1*, *2*). Effects of viscosity and compressibility on the bubble dynamics have been investigated (*3*, *4*). Barlow and Langlois (*5*) considered the diffusion-fed growth of a spherical gas bubble into a Newtonian viscous liquid under isothermal conditions. Later, the problem of isothermal bubble growth, dominated by viscosity and diffusion, was studied (*6*).

In problems of heat transfer with boiling, the time history of bubble formation and growth in a superheated liquid is of great importance. For nucleate boiling in a subcooled liquid, the collapse of bubbles must be considered in addition to the formation and growth. The so-called *extended Rayleigh's equation* (*7*) has been solved for the growth of a vapor bubble in a superheated liquid (*8*, *9*). The dynamics of bubble in binary mixtures are treated in references *10* and *11*. Recently, the mechanics of vapor bubble collapse under spherically symmetrical conditions were examined to ascertain the relative importance of the liquid inertia and heat transfer rate (*12*).

In this work, the dynamic equations governing the pressure distribution and growth or collapse of a spherical gas or vapor bubble in incompressible fluids are formulated from the conservation laws of mass and momentum. The expression for the rate of energy dissipation, or the rate of irreversible conversion of mechanical energy into internal energy, in the liquid region due to the bubble motion is obtained. The analysis is general and may be applied to both Newtonian and non-Newtonian fluids. Numerical results are obtained for the collapse of the bubble.

ANALYSIS

Consider a spherical bubble growing or collapsing in an infinite mass of homogeneous incompressible liquid. The model of a spherical bubble is justified when its radius is less than 0.5 mm. (*13*). For bubbles of radius between 0.5 and 4.0 mm., which are oblate spheroid approximately, the bubble size may be specified by its equivalent spherical radius. The spherical shape of the bubble remains stable under the action of surface tension in a spherically symmetric external pressure field. The assumptions of the symmetric pressure field and the absence of wall effects and external temperature gradients are reasonable for the bubble growing or collapsing outside a thin thermal boundary layer of the wall. In case of Newtonian fluids, however, the theoretical prediction by the present model is reported in good agreement with experimental results obtained for bubbles growing on a heated surface (*8* to *10*). In the absence of body force and in laminar flow regime (including creeping flow), the equations of continuity and motion in the liquid with constant density may be expressed in spherical coordinates as (*14*)

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) = 0 \tag{1}$$

and

$$\rho_l\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r}\right) = -\frac{\partial p}{\partial r} - |\nabla \cdot \bar{\bar{\tau}}|_r \tag{2}$$